The representations of quivers of type $\mathbb{A}_n$. A fast approach.

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It is well-known that a quiver Q of type $\mathbb{A}_n$ is representation-finite, and that its indecomposable representations are thin (all Jordan-Hölder multiplicities are 0 or 1). By now, various methods of proof are known. The aim of this note is to provide a straightforward arrangement of possible arguments in order to avoid indices and clumsy inductive considerations, but also avoiding somewhat fancy tools such as the Bernstein-Gelfand-Ponomarev reflection functors or bilinear forms and root systems. The proof we present deals with representations in general, not only finite-dimensional ones. We only will use first year linear algebra, namely the existence of bases of vector spaces V, W compatible with a given linear map $V \to W$, and the existence of a basis of a vector space which is compatible with two given subspaces.

Theorem. *Any representation of a quiver of type $\mathbb{A}_n$ is a direct sum of thin representations.*

Proof. If $n = 2$, then we deal with a linear map $f\colon V \to W$. Any first year linear algebra course shows how to obtain a direct decomposition: Take a basis $\mathcal{B}$ of the kernel of f, extend it by a family $\mathcal{B}'$ to a basis of V. Now $\{f(b') \mid b' \in \mathcal{B}'\}$ is a basis of the image $f(V)$ and we extend it by a family $\mathcal{B}''$ to a basis of W.

Thus, let $n \ge 3$ and use induction. We deal with a quiver Q with underlying graph

$$\underset{1}{\circ} \text{---} \underset{2}{\circ} \text{---} \underset{3}{\circ} \text{--} \quad \cdots \quad \text{--} \underset{n-1}{\circ} \text{---} \underset{n}{\circ}$$

Let M be a representation of Q and x a vertex of Q. We call x a *peak for M* provided for any arrow $\alpha\colon y \to z$ the map M_α is injective in case $d(x,z) = d(x,y) - 1$, and surjective in case $d(x,z) = d(x,y) + 1$. Obviously, a thin indecomposable representation M of Q with $M_x \neq 0$ has x as a peak. Second, a direct sum of modules with x as a peak, has x as a peak. And third, if x is a peak for M, then also for any direct summand of M.

We assume now by induction that all representations of quivers of type $\mathbb{A}_{n-1}$ are direct sums of thin representations. We first show:

(1) *Given a vertex $1 < x < n$ of Q, then any representation A of Q can be written as a direct sum $A = B \oplus C \oplus D$, where B has x as a peak, the support of C is contained in $\{1, 2, \dots, x-1\}$ and the support of D is contained in $\{x+1, x+2, \dots, n\}$.*

Proof: We first look at the restriction A' of A to the subquiver Q' with vertices $1, 2, \dots, x$. By assumption, we write A' as a direct sum of thin indecomposable modules, say $A' = B' \oplus C'$, where B' is a direct sum of thin indecomposable representations of Q' with x a peak and C' a direct sum of thin indecomposable representations of Q' with $C'_x = 0$. Second, we look at the restriction A'' of A to the subquiver Q'' with vertices $x, x+1, \dots, n$. Again by assumption, we write A'' as a direct sum of thin indecomposable modules, say $A'' = B'' \oplus D''$, where B'' is a direct sum of thin indecomposable representations of Q'' with x a peak and D'' a direct sum of thin indecomposable representations of Q'' with $D''_x = 0$. Since $C'_x = 0 = D''_x$, we see that $A_x = B'_x = B''_x$. Let B be the subrepresentation

of A defined as follows: Its restriction to Q' is B', its restriction of Q'' is B''. Let C be the subrepresentation of A whose restriction to Q' is C' and $C_y = 0$ for $y \geq x$. Let D be the subrepresentation of A whose restriction to Q'' is D'' and $D_y = 0$ for $y \leq x$. Then clearly $A = B \oplus C \oplus D$, with B having x as a peak, the support of C is contained in $\{1, 2, \dots, x-1\}$ and the support of D is contained in $\{x+1, x+2, \dots, n\}$.

Using (1) twice, first for $x = 2$, then for $x = n-1$, we see:

(2) *Any representation A of Q can be written as a direct sum $A = B \oplus C \oplus D \oplus E$, where B has both 2 and $n-1$ as peaks, the support of C is contained in $\{1\}$, the support of D is contained in $\{3, 4, \dots, n-2\}$ and the support of E is contained in $\{n\}$.*

Since by induction the representations C, D, E are direct sums of thin representations, we may assume that $A = B$, thus that A has both 2 and $n-1$ as peaks. But if 2 and $n-1$ both are peaks, all the maps A_α with α an arrow in-between 2 and $n-1$ are isomorphisms, thus, A is isomorphic to a representation where all these maps A_α are identity maps. It remains to look at the case $n = 3$ and a representation with peak 2. We have to show:

Let Q be a quiver of type $\mathbb{A}_3$ with graph $1-2-3$. Any representation of Q with peak 2 is the direct sum of thin representations. Three different orientations have to be discussed:

$$\underset{1}{\circ} \xrightarrow{\alpha} \underset{2}{\circ} \xleftarrow{\beta} \underset{3}{\circ} \qquad \underset{1}{\circ} \xrightarrow{\alpha} \underset{2}{\circ} \xrightarrow{\beta} \underset{3}{\circ} \qquad \underset{1}{\circ} \xleftarrow{\alpha} \underset{2}{\circ} \xrightarrow{\beta} \underset{3}{\circ}$$

In the first case, we deal with a representation A such that both A_α and A_β are injective, thus up to isomorphism we may assume that A_α and A_β are inclusions of subspaces. Again any first year linear algebra course shows how to obtain a direct decomposition: Take a basis $\mathcal{B}$ of the intersection $A_1 \cap A_3$, extend it by a family $\mathcal{B}'$ to a basis of A_1 and by a family $\mathcal{B}''$ to a basis of A_3. Then the disjoint union $\mathcal{B} \cup \mathcal{B}' \cup \mathcal{B}''$ is a basis of $A_1 + A_2$ and we can extend this by a family $\mathcal{B}'''$ to obtain a basis of A_2.

In the second case, we deal with a representation A such that A_α is injective, A_β is surjective. Thus up to isomorphism we may assume that A_α is the inclusion of a subspace and we denote by A_3' the kernel of A_β. Similar to the first case, we construct a basis of A_2 which is compatible with the two subspaces A_1 and A_3', this yields the required direct decomposition of A.

Finally, in the third case, we deal with a representation A such that both A_α and A_β are surjective. Let A_1' be the kernel of A_α and A_3' the kernel of A_β. Again, as before, we construct a basis of A_2 which is compatible with the two subspaces A_1' and A_3'. This completes the proof.

Of course, as a consequence we obtain: *Given two filtrations $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_m$ and $U_1' \subseteq U_2' \subseteq \cdots \subseteq U_{m'}'$ of a vector space V, there is a basis of V which is compatible with all these subspaces U_i, U_j'.* Namely, look at the corresponding representation A of the following quiver of type $\mathbb{A}_{m+m'+1}$

$$\underset{1}{\circ} \longrightarrow \underset{2}{\circ} \longrightarrow \cdots \longrightarrow \underset{m}{\circ} \longrightarrow \underset{\omega}{\circ} \longleftarrow \underset{m'}{\circ} \longleftarrow \cdots \longleftarrow \underset{2'}{\circ} \longleftarrow \underset{1'}{\circ}$$

with $A_\omega = V$, $A_i = U_i$ for $1 \leq i \leq m$ and $A_{j'} = U_j'$ for $1 \leq j \leq m'$.

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